

Friday, October 10, Lecture 18: Contour Integration, Cauchy's Theorems

Last time... Contour integration

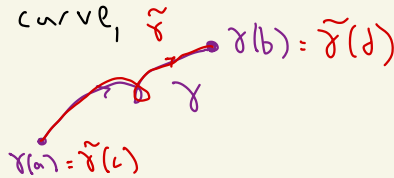
$$\gamma: [a, b] \rightarrow \mathbb{C}, \quad f: \mathbb{C} \rightarrow \mathbb{C}$$

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$$



Fact: (see Proreparameterization in last lecture)

If $\gamma: [a, b] \rightarrow \mathbb{C}$ and $\tilde{\gamma}: [c, d] \rightarrow \mathbb{C}$ trace out the same oriented curve,



$$\int_{\gamma} f = \int_{\tilde{\gamma}} f$$

for any continuous $f: \mathbb{C} \rightarrow \mathbb{C}$

Another option for contour integrals: differential forms.

• Just like vector calc, we can express it as

$$dz = d(x+iy) = dx + i dy$$

$$\Rightarrow f(z) dz = (u+iv)(dx+idy) = u dx + i v dx + i u dy - v dy \\ = (u dx - v dy) + i(u dy + v dx)$$

Prop

If $f(z) = u(x,y) + i v(x,y)$, then

$$\int_{\gamma} f = \int_{\gamma} u dx - v dy + i \int_{\gamma} u dy + v dx$$

$$\text{PF: } \gamma(t) = x(t) + i y(t).$$

$$f'(\gamma(t)) \cdot \gamma'(t) = (u(x(t), y(t)) + i v(x(t), y(t))) \cdot (x'(t) + i y'(t))$$

Then integrate both sides. $\square$

The usual integral properties apply.

• of course, linearity: if $c_1, c_2 \in \mathbb{C}$,

$$\int_{\gamma} c_1 f + c_2 g = c_1 \int_{\gamma} f + c_2 \int_{\gamma} g.$$

• Useful Quantity: Arc length

$$l(\gamma) = \int_a^b |\gamma'(t)| dt$$

$$|\gamma'(t)| = |(x'(t), y'(t))|$$

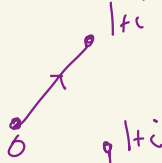
$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Ex: Compute

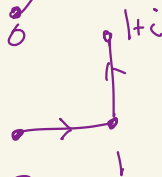
$$\int_{\gamma} z^2 dz$$

along

$$\gamma_1 =$$



$$\gamma_2 =$$



$$\gamma_1: \gamma_1(t) = t(1+i) + (1-t) \cdot 0 = t(1+i) \\ t \in [0, 1]$$

$$\gamma_1'(t)$$

$$\begin{aligned} \text{so } \int_{\gamma_1} z^2 dz &= \int_0^1 (t(1+i))^2 \overbrace{(1+i)}^{\gamma_1'(t)} dt \\ &= \int_0^1 (t^2(1+2i-1))(1+i) dt \\ &= 2i(1+i) \int_0^1 t^2 dt \\ &= 2i(1+i) \left(\frac{1}{3} - 0 \right) \\ &= \frac{2}{3}(i-1) \end{aligned}$$

$$\begin{aligned}
 \gamma_2: \int_{\gamma_2} z^2 dz &= \int_0^1 (x+i0)^2 dx + \int_0^1 (1+iy)^2 i dy \\
 &= \frac{1}{3}(1^3 - 0^3) + \int_0^1 (1+2iy-y^2) i dy \\
 &= \frac{1}{3} + i \int_0^1 (1-y^2) dy - \int_0^1 2y dy \\
 &= \frac{1}{3} + i(1 - \frac{1}{3}1^3) - 1 \\
 &= -\frac{2}{3} + i(\frac{2}{3}) \\
 &= \frac{2}{3}(i-1)
 \end{aligned}$$

Thm: Fundamental Thm of Complex Line Integrals

If $f = F'$, then if $\gamma: [0,1] \rightarrow \mathbb{C}$

$$\int_{\gamma} f dz = F(\gamma(1)) - F(\gamma(0))$$

Pf: Hw 7.

* Compare: line integrals of conservative vector fields.

Notice that this implies path independence.

$\gamma(0) = \tilde{\gamma}(0)$ and $\gamma(1) = \tilde{\gamma}(1)$ and $f = F'$

$$\Rightarrow \int_{\gamma} f = \int_{\tilde{\gamma}} f$$

Let's think back to vector calc.

The earlier Prop. reveals: $\int_{\gamma} f$ is

secretly a combination of two line integrals:

if $f(x,y) = u(x,y) + i v(x,y)$, then

$$\begin{aligned}\int_{\gamma} f dz &= \int_{\gamma} (u+iv)(dx+idy) \\ &= \int_{\gamma} u dx - v dy + i \int_{\gamma} u dy + v dx.\end{aligned}$$

Hmm... maybe we can use theorems from vector calc to say something.

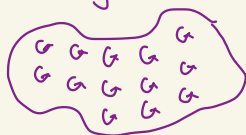
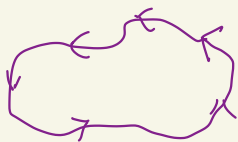
Recall... Green's Theorem.

- γ - simple, cts, positively oriented, smooth loop in $\mathbb{R}^2$.

- Ω - simply connected region bounded by γ

$g = (P, Q)$ be a vector field ^(ctsly diff.) (so $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$,
 $P, Q: \mathbb{R}^2 \rightarrow \mathbb{R}$).

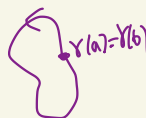
Then
$$\int_{\gamma} P dx + Q dy = \iint_{\Omega} \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)}_{\text{curl } g} dx dy$$



γ simple = does not cross itself
 Ω simply connected = no holes!

• Loop =

$\gamma(a) = \gamma(b)$



So:
$$\int_{\gamma} f dz = \underbrace{\int_{\gamma} u dx - v dy}_{(1)} + i \underbrace{\int_{\gamma} u dy + v dx}_{(2)}$$

Let's assume f is continuously differentiable.

① Green's thm: Pick $P=u, Q=-v$.

$$\int_{\gamma} u dx + (-v) dy = \iint_{\Omega} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy$$

$$\begin{aligned} \text{CR eqns: } \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \\ &= \iint_{\Omega} \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) dx dy \\ &= 0 \end{aligned}$$

② Green's thm: Pick $P=v, Q=u$.

$$\int_{\gamma} v dx + u dy = \iint_{\Omega} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy$$

$$\text{CR eqns: } \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = 0$$

Thm (Cauchy's thm)

Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic on a simply connected region Ω , and let $\gamma: [a,b] \rightarrow \Omega$ be a cts, piecewise smooth loop. Then

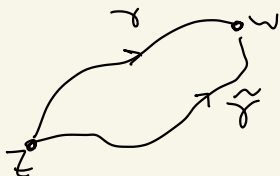
$$\int_{\gamma} f(z) dz = 0.$$

Corollary:

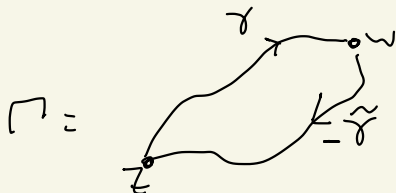
If Ω simply connected, line integrals of f are path-independent.

pf:

Let $z, w \in \Omega$ be endpoints. Take two paths, $\gamma, \tilde{\gamma} \subseteq \Omega$.



Then observe that $\Gamma := \gamma - \tilde{\gamma}$ is a loop:



$\Rightarrow$ By Cauchy's theorem,

$$0 = \int_{\Gamma} f(z) dz = \int_{\gamma - \tilde{\gamma}} f(z) dz = \int_{\gamma} f(z) dz - \int_{\tilde{\gamma}} f(z) dz$$

so

$$\int_{\tilde{\gamma}} f(z) dz = \int_{\gamma} f(z) dz.$$



The simply connected assumption on Ω is crucial!

Ex: (extremely important).

Let $f(z) = \frac{1}{z}$, and $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.

Then

$$\begin{aligned} \int_{\gamma} \frac{1}{z} dz &= \int_0^{2\pi} \frac{1}{(e^{it})} \cdot i e^{it} dt \\ &= \int_0^{2\pi} i dt \\ &= 2\pi i. \end{aligned}$$

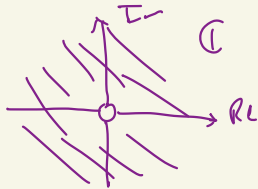


Why doesn't Cauchy's thm apply?

$\frac{1}{z}$ has a singularity at $z=0$, so the domain Ω

has a hole and is not simply connected:

$$\Omega = \mathbb{C} \setminus \{0\},$$



HW: Primer to residue thm.

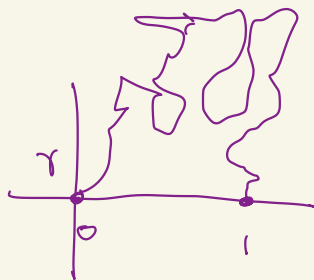
This is a surprisingly powerful theorem.

We may use it to "deform" line integrals.

Ex: Deformation

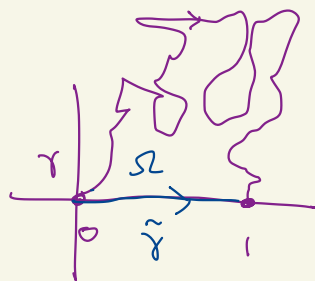
Compute

$\int_{\gamma} e^z dz$ along



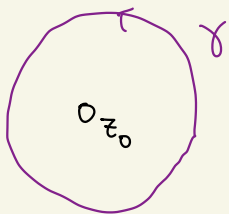
By Cauchy's thm, since e^z holomorphic inside Ω ,

$$\begin{aligned}\int_{\gamma} e^z dz &= \int_{\tilde{\gamma}} e^z dz \\ &= \int_0^1 e^x dx \\ &= e^1 - 1.\end{aligned}$$



Cauchy's integral formula

How to handle holes? well...

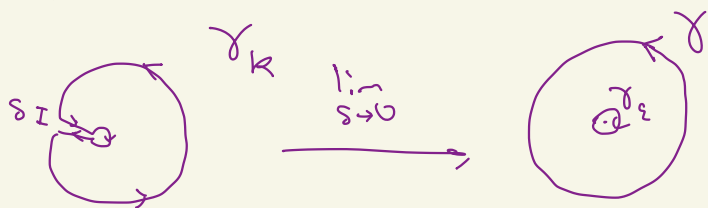


We can't immediately

calculate $\int_{\gamma} \frac{f(z)}{z-z_0} dz$,

since $\Omega = \mathbb{C} \setminus \{z_0\}$ is not simply connected.

But...



Cauchy

$$0 = \int_{\gamma_k} \frac{f(z)}{z-z_0} dz = \int_{\gamma} \frac{f(z)}{z-z_0} dz + \underbrace{\int_{\gamma_\epsilon} \frac{f(z)}{z-z_0} dz}_{\text{Cauchy}}$$

Since f holomorphic, f is continuous, so as $z \rightarrow 0$, $f(z) \approx f(z_0)$ for all z inside γ_ϵ .

$$\approx \int_{\gamma} \frac{f(z)}{z-z_0} dz + f(z_0) \underbrace{\int_{\gamma_\epsilon} \frac{1}{z-z_0} dz}_{\text{Cauchy}}$$

This is just like $\int \frac{1}{z}$, but offset and backwards.

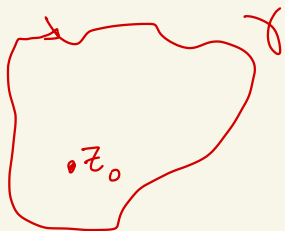


$$= \int_{\gamma} \frac{f(z)}{z-z_0} dz + f(z_0)(-2\pi i)$$

Thm: Cauchy's Integral Formula

Let f be holomorphic on simply connected Ω and γ a loop in Ω .
Let z_0 be inside γ . Then

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz$$



$f(z_0)$ is completely
determined by
its boundary values!

How crazy is that??